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A DISCUSSION OF CRITICAL PARAMETERS WHICH CAN OCCUR IN FRICTIONALLY HEATED NON-NEWTONIAN FLUID FLOWS

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Abstract—Fully developed steady flows of purely viscous non-Newtonian fluids with viscous heat generations are usually governed by non-linear Sturm–Liouville differential system. It is shown that under certain conditions that the non-linear eigenvalues of the system take a critical value above which no local solutions are obtainable. The theory is applied to specific flow situations and an upper bound for this critical parameter is evaluated in each case. It is shown that the non-linear heat generation term is responsible for double valued pressure gradient flow rate characteristics in the models of screw extruder systems. Computed values of the upper bound of the critical parameter are given for a specific model of a screw extruder.

NOMENCLATURE

t_{ij},	Cartesian components of the stress tensor;
C_0, s, b, n,	material parameters;
e_{ij},	Cartesian components of the rate of deformation tensor;
I_2,	second invariant of e_{ij};
p,	isotropic pressure;
T,	temperature;
T_0,	reference temperature;
K,	thermal conductivity;
x, y, z,	Cartesian coordinates in flat plate problems;
r, θ, z,	cylindrical polar coordinates;
v_x, v_y, v_z,	Cartesian components of the velocity vector;
v_θ, v_r, v_z,	cylindrical polar components of the velocity vector;
h,	gap between flat plates;
R,	radius of right cylinder;
X, Y, Z,	dimensionless coordinates;
x,	dimensionless radius in pipe problem;
W, U, U_x, U_y,	dimensionless velocity components;
ϕ,	dimensionless temperature;
τ^*,	constant shear stress in flat plate problem;
λ,	dimensionless shear stress or pressure gradient;
Λ,	parameter associated with λ obtained from the linear comparison system;
Y_1, Y_2,	zero stress levels;
J_0,	zeroth order Bessel Function;
P_1,	pressure gradient ratio;
Q,	flow rate.

All other minor symbols are defined in the text.

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INTRODUCTION

It is well known that frictional heating effects are an important feature of many polymer melt flows. Polymer melts can frequently be modelled by purely viscous non-Newtonian fluids. Reports of critical parameters occurring in non-isothermal flows of purely viscous (Newtonian and non-Newtonian) fluids can be found in the literature, for example, Joseph [1], Joseph [2], Gruntfest [12], Turian [14], Winter [22]. For some viscometric flows, Couette and Poiseuille, exact solutions of the governing non-linear ordinary differential equations have been developed (see Kearsley [8], Gavis and Lawrence [9, 10], Martin [13], Turian [14]) and exact values of the critical parameters, above which no steady solutions exist, can be found. Other more complicated models, Colwell and Nickolls [16] Zamodits and Pearson [17] have no closed form solutions and numerical computation is necessary. For these flows if the viscosity of the fluid is temperature dependent, critical values of pressure gradient (shear stress) parameters can occur, above which no steady flows exist. The non-linear heat source can be shown to be directly responsible for the double valued flow rate-pressure gradient characteristics, which have been exhibited in the literature. Joseph [2] has indicated how a close upper bound to the critical parameter of a non-linear system can be evaluated from an associated linear system. Cohen [23] has developed similar methods for chemical reactors.

Section 2 gives the fluid model used in the paper. Section 3 is devoted to retrieving the exact values of critical parameters which occur in some viscometric flows and comparing them with the upper bounds obtained from the associated linear systems. This should give some confidence in the accuracy of the estimated critical parameter when no closed form solution can be found.

Section 4 gives results for the duct model of a single screw extruder used by Zamodits and Pearson [17]. Some numerical values for the upper bound of the pressure gradient parameter are given.

A brief account of the computational techniques used in solving the linear Sturm–Liouville system is given in the Appendix.

1. DISCUSSION OF THE GENERAL GOVERNING SECOND ORDER DIFFERENTIAL EQUATION

The non-dimensionalized energy equation occuring in the steady flow of incompressible viscous fluids heated by internal friction takes the general form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right] + \lambda f(x)\,\phi(\psi) = 0 \qquad (1.1)$$

in some interval $a < x < b$, where $a, b \geqslant 0$ and λ is a non-negative parameter. Joseph in a number of papers [1–5] has discussed a more specific case

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \lambda\phi(\psi) = 0 \qquad (1.2)$$

with specific boundary conditions,

$$\frac{\mathrm{d}\psi}{\mathrm{d}x}(0) = \psi(1) = 0. \qquad (1.3)$$

Here we shall endeavour to extend this theory to deal with equation (1.1) and the general homogeneous boundary conditions

$$\begin{aligned} \beta(a)\frac{\mathrm{d}\psi}{\mathrm{d}x}(a) + \gamma(a)\,\psi(a) &= 0 \\ \beta(b)\frac{\mathrm{d}\psi}{\mathrm{d}x}(b) + \gamma(b)\,\psi(b) &= 0. \end{aligned} \qquad (1.4)$$

Systems similar to (1.1) with (1.4) have received attention, with a different emphasis, from a number of authors, for example, Cohen [23, 24], Keller and Cohen [25], Dean and Chambré [26].

It will be shown that under certain assumptions there exists a value λ_{crit} of λ such that locally no solutions of (1.1) subject to (1.4) exist for $\lambda > \lambda_{\text{crit}}$ while at least two solutions exist for $\lambda < \lambda_{\text{crit}}$.

We shall consider the set of problems for which $p(x)$, $f(x)$, $\psi(x)$, $\phi(\psi)$ are functions of at least class $C^{(2)}$ on $a < x < b$, such that $p(x) > 0$, is monotonically increasing, and $f(x) > 0$, $\psi(x) > 0$ in $a < x < b$. Further it is assumed that $\phi(\psi)$ is a monotonically increasing function of ψ such that $\phi(\psi) \geqslant 1$ and $\phi(0) = 1$.

From the above assumptions it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\psi}{\mathrm{d}x}\right] \leqslant 0 \qquad \text{in } a < x < b. \quad (1.5)$$

Case 1

We shall consider $d\psi/dx > 0$ at $x = a$, and $d\psi/dx < 0$ at $x = b$; these conditions correspond to heat flowing out of the system in the viscous heating problem. With the above conditions there is certainly at least one turning point because $d\psi/dx$ changes sign. Neglecting the trivial case $\lambda = 0$ equation (1.1) implies that

$$\frac{d}{dx}\left(p(x)\frac{d\psi}{dx}\right) < 0 \qquad \text{in } (a, b).$$

Consequently $p(x)d\psi/dx$ is monotonically decreasing in (a, b). Since $d\psi/dx$ changes sign in the interval and $p(x) > 0$ throughout the interval this implies that there is one stationary point in (a, b) and this must be a maximum. With the added restriction of increasing monotonicity of $p(x)$, $d\psi/dx$ will be monotonically decreasing in (a, b).

Case 2

If $d\psi/dx = 0$ at one end point (corresponding to an adiabatic heat boundary condition), then there is certainly a stationary point, and since $d\psi/dx$ is monotonically decreasing in (a, b) there can be no other stationary point in (a, b), so that the maximum occurs at either $x = a$ or $x = b$.

These results imply that a ψ_{max} always exists for a given set of boundary conditions and a given λ, in the same way one can consider λ as a function of ψ_{max}. It may be that ψ_{max} occurs at different values of x in the interval as λ changes although the boundary conditions remain the same.

It will be assumed that λ is a continuously differentiable function of ψ_{max} to whatever order we require.

Integrating equation (1.1) gives

$$\psi = \lambda \int_x^b \frac{d\eta}{p(\eta)} \int_a^\eta f(\gamma)\phi\{\psi(\gamma)\}\, d\gamma. \qquad (1.6)$$

Since $d\psi/dx$ is a monotonically decreasing function

$$(\psi_m - \psi(b))\left(\frac{b - x}{b - x_0}\right) \leqslant \psi \leqslant \psi_m \qquad \text{for } x_0 \leqslant x \leqslant b, \qquad (1.7)$$

where $\psi(x_0) = \psi_{max} = \psi_m$.

The monotonicity of ϕ implies that

$$\phi\left\{(\psi_m - \psi(b))\left(\frac{b - x}{b - x_0}\right)\right\} \phi(\psi) \leqslant \phi(\psi_m). \qquad (1.8)$$

Consequently equations (1.8), (1.6) and (1.7) imply that

$$\lambda \int_x^b \frac{d\eta}{p(\eta)} \int_{x_0}^\eta f(\gamma)\phi\left\{(\psi_m - \psi(b))\left(\frac{b - \gamma}{b - x_0}\right)\right\} \times d\gamma \leqslant \psi \leqslant \lambda\phi(\psi_m) \int_x^b \frac{1}{p(\eta)} \int_{x_0}^\eta f(\gamma)\, d\gamma. \qquad (1.9)$$

In particular

$$\lambda \int_x^b \frac{d\eta}{p(\eta)} \int_{x_0}^\eta f(\gamma)\phi\left\{(\psi_m - \psi(b))\left(\frac{b - \gamma}{b - x_0}\right)\right\} \times d\gamma \leqslant \psi_{max} \leqslant \lambda\phi(\psi_m) \int_{x_0}^b \frac{1}{p(\eta)} \int_{x_0}^\eta f(\gamma)\, d\gamma \qquad (1.10)$$

which gives

$$\frac{G\psi_m}{\phi(\psi_m)} \leqslant \lambda \leqslant \psi_m\left[\int_{x_0}^b \frac{d\eta}{p(\eta)} \int_{x_0}^\eta f(\gamma)\phi \times \left\{(\psi_m - \psi(b))\left(\frac{b - \gamma}{b - x_0}\right)\right\} d\gamma\right]^{-1}, \qquad (1.11)$$

where

$$G = \left[\int_{x_0}^b \frac{1}{p(\eta)} \int_{x_0}^\eta f(\gamma)\, d\gamma\right]^{-1} > 0.$$

The inequality (1.11) can be regarded as providing functions of ψ_m which are upper and lower bounds of λ considered as a function of ψ_m.

The behaviour of the solutions of equation (1.1) depends, in the main, on the order with which

$\phi(\psi_m)$ increases with ψ_m, as ψ_m tends to infinity. We shall assume that the asymptotic development of ϕ is of the form.

$$\lim_{\psi_m \to \infty} \phi[\psi_m] \to A\psi_m^k. \tag{1.12}$$

Three cases can be considered with this asymptotic development of ϕ, namely, $0 \leqslant k < 1$, $k = 1$, $k > 1$ the case of interest as far as the later work is concerned being when $k > 1$. For many viscous heating problems $\phi(\psi) = e^{\beta\psi}$, with β a positive constant.

In this asymptotic limit

$$\frac{G}{A}\frac{\psi_m}{\psi_m^k} \leqslant \lambda \leqslant \frac{\psi_m \tilde{G}}{A(\psi_m - \psi(b))^k}, \tag{1.13}$$

where

$$\tilde{G} = \left[\int_{x_0}^{b} \frac{d\eta}{p(\eta)} \int_{x_0}^{\eta} f(\gamma)\left(\frac{b-\gamma}{b-x_0}\right)^k d\gamma\right]^{-1} > 0.$$

It can be seen from equation (1.13) that as $\psi_m \to \infty$, $\lambda \to 0$. This implies that the parameter is not a unique function of ψ_m. Both the upper and lower bounds for λ given by (1.11) possess two zeros and must, if continuous functions of ψ_m, each possess at least one maximum. Since (1.11) implies that no solutions of equation (1.1) exist when λ exceeds the maximum value of the upper bound there must exist a value $\lambda = \lambda_{\text{crit}}$ above which no solution to (1.1) can be found, and below which at least two solutions can exist. The above statement applies only to values of λ in a neighbourhood of λ_{crit}. Existence of solutions has been discussed by Reginer [6] and Kaganov [7].

In the special case when $d\psi/dx = 0$ at one end point say $x = a$ for definiteness, x_0 is replaced by a and equation (1.7) becomes

$$\{\psi_m - \psi(b)\}\left(\frac{b-x}{b-a}\right) \leqslant \psi \leqslant \psi_m \text{ for } a \leqslant x \leqslant b,$$

and (1.11) becomes

$$\frac{G\psi_m}{\phi(\psi_m)} \leqslant \lambda \leqslant \psi_m\left[\int_a^b \frac{d\eta}{p(\eta)} \int_a^\eta f(\gamma)\phi \times \left\{(\psi_m - \psi_0)\left(\frac{b-\gamma}{b-a}\right)\right\} d\gamma\right]^{-1},$$

with

$$G = \left[\int_a^b \frac{1}{p(\eta)} \int_a^\eta f(\gamma)\, d\gamma\right]^{-1} > 0.$$

Using an argument similar to Joseph [1] we can now show that the first stationary point of $\lambda(\psi_m)$ is maximum when $d^2\phi/d\psi^2 = \phi'' > 0$. Consider ψ as a function of the parameter ψ_m and the variable x, and write

$$\dot{\psi} = \frac{\partial\psi}{\partial\psi_m}.$$

Equation (1.1) can be thought of as identity in ψ_m in the sense that it is true whatever ψ_m is developed by specifying a value of $\lambda \leqslant \lambda_{\text{crit}}$.

Differentiating equations (1.1) and (1.4) gives

$$\frac{d}{dx}\left[p(x)\frac{d\dot{\psi}}{dx}\right] + \dot{\lambda}f(x)\phi + \lambda f(x)\phi'\dot{\psi} = 0, \tag{1.14}$$

$$\frac{d}{dx}\left[p(x)\frac{d\ddot{\psi}}{dx}\right] + \ddot{\lambda}f(x)\phi + 2\dot{\lambda}f(x)\phi'\dot{\psi} + f(x)\lambda\phi''\dot{\psi}^2 + \lambda f(x)\phi'\ddot{\psi} = 0, \tag{1.15}$$

with

$$\beta(a)\frac{d\dot{\psi}}{dx}(a) + \gamma(a)\dot{\psi}(a) = \beta(b)\frac{d\dot{\psi}}{dx}(b) + \gamma(b)\dot{\psi}(b) = 0, \tag{1.16}$$

and

$$\beta(a)\frac{d\ddot{\psi}}{dx}(a) + \gamma(a)\ddot{\psi}(a) = \beta(b)\frac{d\ddot{\psi}}{dx}(b) + \gamma(b)\ddot{\psi}(b) = 0. \tag{1.17}$$

Multiplying equation (1.14) by $\ddot{\psi}$ and (1.15) by $\dot{\psi}$ and integrating over (a, b) gives

$$\left[p(x)\ddot{\psi}\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x}\right]_a^b - \int_a^b p(x)\frac{\mathrm{d}\ddot{\psi}}{\mathrm{d}x}\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x}\,\mathrm{d}x + \dot{\lambda}\int_a^b$$

$$\times\; f(x)\phi\ddot{\psi}\,\mathrm{d}x + \dot{\lambda}\int_a^b f(x)\phi'\dot{\psi}\ddot{\psi}\,\mathrm{d}x = 0, \qquad (1.18)$$

$$\left[p(x)\ddot{\psi}\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x}\right]_a^b - \int_a^b p(x)\frac{\mathrm{d}\ddot{\psi}}{\mathrm{d}x}\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x}\,\mathrm{d}x + \ddot{\lambda}\int_a^b$$

$$\times\; f(x)\phi\dot{\psi}\,\mathrm{d}x + 2\dot{\lambda}\int_a^b f(x)\phi'\dot{\psi}^2\,\mathrm{d}x$$

$$+\,\lambda\int_a^b f(x)\phi''(\dot{\psi})^3\,\mathrm{d}x + \lambda\int_a^b f(x)\phi'\dot{\psi}\ddot{\psi}\,\mathrm{d}x = 0. \qquad (1.19)$$

Subtracting equation (1.19) from (1.18) and using (1.16) and (1.17) gives

$$\ddot{\lambda} = \frac{-\lambda\int_a^b f(x)\phi''\dot{\psi}^3\,\mathrm{d}x}{\int_a^b f(x)\phi\dot{\psi}\,\mathrm{d}x} \qquad (1.20)$$

when $\dot{\lambda} = 0$. The functions f, ϕ, ϕ'' are all positive so that the sign of $\ddot{\lambda}$ depends simply on the function $\dot{\psi}$. If $\dot{\psi}$ takes the same sign throughout (a, b) then $\ddot{\lambda}$ is negative and the turning point is a local maximum. We shall show using a continuity argument that $\dot{\psi}$ must be positive throughout (a, b) when $\dot{\lambda}$ first vanishes.

It was assumed earlier that $\mathrm{d}\psi/\mathrm{d}x > 0$ at $x = a$ and $\mathrm{d}\psi/\mathrm{d}x < 0$ at $x = b$, assuming $\psi \geqslant 0$ implies that

$$\frac{\gamma(a)}{\beta(a)} \leqslant 0 \text{ and } \frac{\gamma(b)}{\beta(b)} \geqslant 0.$$

Equation (1.14) shows that as $\lambda \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\dot{\psi}}{\mathrm{d}x}\right] < 0,$$

so that in this limit $\mathrm{d}\dot{\psi}/\mathrm{d}x$ is monotonically decreasing throughout. We shall now consider the various distributions available to $\dot{\psi}$ in the interval $[a, b]$.

Case 1. $\mathrm{d}\dot{\psi}/\mathrm{d}x = 0$ *at an interior point of* $[a, b]$.

Suppose $\dot{\psi} < 0$ at $x = a$, then necessarily $\mathrm{d}\dot{\psi}/\mathrm{d}x \leqslant 0$ from the boundary conditions. So that in the limit $\lambda \to 0$, $\mathrm{d}\dot{\psi}/\mathrm{d}x$ is a monotonically decreasing function, which implies that $\dot{\psi}$ is never positive. However, if $\psi = \psi_m$ at $x = x_0$, then

$$\psi(x_0, \psi_m) = \psi_m,$$

which upon differentiation with respect to ψ_m gives

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right)_{x=x_0}\frac{\partial x_0}{\partial\psi_m} + \dot{\psi}_{x=x_0} = 1.$$

But $\mathrm{d}\psi/\mathrm{d}x = 0$ when ψ takes it maximum value, therefore

$$\dot{\psi}_{x=x_0} = 1$$

whatever the value of ψ_m.

This result implies that $\dot{\psi} < 0$ at $x = a$ is impossible, in the limiting case $\lambda \to 0$.

It migh be possible, however, for $\dot{\psi} > 0$ in the limiting case and for $\dot{\psi}$ to decrease to a negative value as λ increases. However, as $\dot{\psi} \to 0$, $\mathrm{d}\dot{\psi}/\mathrm{d}x \to 0$ so that in this limiting case the curve would again never take positive values and the condition $\dot{\psi} = 1$ at some $x_0 \in (a, b)$ would be violated. Using this continuity argument it follows that $\dot{\psi} > 0$ at $x = a$ for all λ up to the first zero in its derivative $\mathrm{d}\lambda/\mathrm{d}\phi_{\max}$. This implies that $\dot{\psi} > 0$, $\mathrm{d}\dot{\psi}/\mathrm{d}x > 0$ in a neighbourhood of $x = a$, for all λ, at least until $\mathrm{d}\lambda/\mathrm{d}\phi_{\max} = 0$. A similar argument is used at $x = b$. Suppose $\dot{\psi} < 0$ at $x = b$, then $\mathrm{d}\dot{\psi}/\mathrm{d}x > 0$ at $x = b$. In the limiting case $\lambda \to 0$ this would violate the condition that $\mathrm{d}\dot{\psi}/\mathrm{d}x$ is monotonically decreasing, since there must be at least two turning points if $\mathrm{d}\dot{\psi}/\mathrm{d}x > 0$ at $x = a$ and $\mathrm{d}\dot{\psi}/\mathrm{d}x > 0$ at $x = b$. Hence, $\dot{\psi} > 0$ is limiting case $\lambda \to 0$. Again it may be possible for $\dot{\psi}$ to become negative as λ increases. However, as $\dot{\psi} \to 0$, $\mathrm{d}\dot{\psi}/\mathrm{d}x \to 0$

at $x = b$, and this would violate the condition of decreasing monotonicity of $d\dot{\psi}/dx$. So $\dot{\psi} > 0$ at $x = b$ for λ increasing, at least until first zero in λ is reached. These results imply that $\dot{\psi} > 0$, $d\dot{\psi}/dx < 0$ in a neighbourhood of $x = b$ at least until $d\lambda/d\phi_{max} = 0$.

It is still possible that $\dot{\psi}$ has a zero in (a, b) for some λ before $d\lambda/d\phi_{max} = 0$. However, if this is so then $\dot{\psi}$ must have at least two zeros and $\dot{\psi}$ must be negative on some subinterval of (a, b) and have a local minimum there. This implies that in some neighbourhood of $x = b$ for positive $\dot{\psi}$, $d\dot{\psi}/dx$ is increasing, which contradicts the decreasing monotonicity condition for positive $\dot{\psi}$. Consequently $\dot{\psi}$ is positive throughout $[a, b]$ and $d\dot{\psi}/dx$ is monotonically decreasing there. (The special case of a double zero at some interior point does not invalidate the argument). Equation (1.20) now implies that at the first zero of $\dot{\lambda}$ the turning point $\lambda_{crit}(\psi_{max})$ is a local maximum.

Case 2. $d\psi/dx = 0$ *at an end point*, $x =$ a *(say)*

A similar argument holds if $d\dot{\psi}/dx = 0$ (corresponds to $d\psi/dx = 0$) at one end point, say $x = a$ for definiteness. Then $\dot{\psi}$ must be positive at $x = a$ for all λ before $\dot{\lambda} = 0$ and consequently $\dot{\psi} > 0$ in the interior of the interval, although it is possible for $\dot{\psi} = 0$ at $x = b$ provided $d\dot{\psi}/dx < 0$.

Case 3. $\dot{\psi} = 0$ *at both end points*

If $\dot{\psi} = 0$ at both end points, then $d\dot{\psi}/dx$ must be monotonically decreasing in a neighbourhood of each end point and by the same argument as in Case 1, $\dot{\psi}$ has no other zeros on the interior of $[a, b]$.

In all cases under consideration there exists a value λ_{crit} of λ such that locally no solutions of (1.1) subject to (1.4) exist for $\lambda > \lambda_{crit}$, while at least two solutions exist for $\lambda < \lambda_{crit}$.

Joseph [2] has shown that the values of $\lambda > 0$ for which (1.1) and (1.4) have positive solutions ψ are bonded above by a composite expression. If equations (1.1) and (1.4) are compared with the linear homogeneous, self adjoint system

$$\frac{d}{dx}\left[p(x)\frac{d\hat{\psi}}{dx}\right] + \Lambda f(x)\hat{\psi} = 0, \tag{1.21}$$

$$\left.\begin{aligned} &\beta(a)\frac{d\hat{\psi}}{dx}(a) + \gamma(a)\hat{\psi}(a) = 0, \\ &\beta(b)\frac{d\hat{\psi}}{dx}(b) + \gamma(b)\hat{\psi}(b) = 0, \end{aligned}\right. \tag{1.22}$$

it can be shown that

$$\frac{\lambda}{\Lambda_0} = \begin{cases} \dfrac{\int_a^b [f\hat{\psi}_0\psi\phi(\psi)/\phi(\psi)]\,dx}{\int_a^b f\hat{\psi}_0\phi(\psi)\,dx} \leqslant \max\limits_{\psi=0}\dfrac{\psi}{\phi(\psi)}, \\ \left\{1 + \dfrac{\int_a^b f\hat{\psi}_0 G(\psi)\,dx}{\int_a^b f\hat{\psi}_0\psi\,dx}\right\}^{-1} \leqslant \dfrac{\psi_m}{\psi_{m+1}}, \end{cases} \tag{1.23}$$

where Λ_0, $\hat{\psi}_0$ are the least eigenvalue and associated eigenfunction of system (1.21) and 1.22). This result implies that

$$\lambda \leqslant \Lambda_0 \min\begin{cases} \max y/\phi(y). \\ \psi_m/\psi_{m+1} \end{cases} \tag{1.24}$$

2. FLUID MODEL

Throughout this paper we shall assume a power law constitutive equation (see Pearson [15]) which expresses the stress, to within an arbitrary isotropic pressure, as a function of the deformation rate tensor and temperature. The constitutive equation can be written in Cartesian tensor notation as

$$t_{ij} = -p\delta_{ij} + 2^{1-2s}C_0\,e^{-b(T-T_0)}(I_2)^{-s}e_{ij} \qquad i, j = 1, 2, 3, \tag{2.1}$$

where t_{ij}, e_{ij} and δ_{ij} are, respectively, the stress tensor, the deformation rate tensor and the Kronecker delta. $I_2 = \frac{1}{2}e_{ij}e_{ij}$ is the second variant of e_{ij}, p the isotropic pressure and T the absolute temperature. C_0, s and b are constants for any given fluid modelled by equation (1.25). When $s = 0$, C_0 becomes equal to the Newtonian viscosity. T_0 is a convenient reference temperature at which C_0 is measured.

3. COUETTE AND POISEUILLE FLOWS

In this section we shall show that for those flows to which closed form solutions exist the upper bound of λ estimated from the linear Sturm–Liouville system (1.21) and (1.22) is accurate to within 5 or 6 per cent. Numerous exact solutions exist in the literature for fluids modelled by a purely viscous constitutive equation, Kearsley [8], Gavis, and Laurence [9, 10], Nihoul [11], Gruntfest [12], Martin [13] and Turian [14].

Plane Couette flow

We shall consider the simple shear flow of a viscous fluid between two flat plates. Referred to a rectangular cartesian coordinate system $Oxyz$ we assume that the lower plate lies in the plane $y = 0$ and the upper plate lies in the plane $y = h$, where z is measured parallel and y at right angles to the plate. The upper plate is assumed to move with a constant velocity V in the z-direction while the lower plate is considered to be at rest. The steady state equations of motion and energy for incompressible fluids neglecting heating by convection are

$$\frac{\mathrm{d}t_{zy}}{\mathrm{d}y} = 0, \tag{3.1}$$

and

$$K\frac{\mathrm{d}^2T}{\mathrm{d}y^2} + t_{zy}\frac{\mathrm{d}v_z}{\mathrm{d}y} = 0. \tag{3.2}$$

The thermal conductivity, K, of the fluid is assumed constant.

The only non-zero stress component is given by

$$t_{zy} = 2^{1-2s}C_0\,\mathrm{e}^{-b(T-T_0)}\left\{\left(\frac{1}{2}\frac{\mathrm{d}v_z}{\mathrm{d}y}\right)^2\right\}^{-s}\frac{1}{2}\frac{\mathrm{d}v_z}{\mathrm{d}y}, \tag{3.3}$$

where $(0, 0, v_z(y))$ is the velocity distribution between the plates. Two sets of boundary conditions will be specifically considered here, although more general homogeneous boundary conditions can easily be discussed. (a) Plate temperatures prescribed and equal.

$$\begin{aligned} v_z &= 0; \quad T = T_0 \text{ when } y = 0, \\ v_z &= U; \quad T = T_0 \text{ when } y = h. \end{aligned} \tag{3.4}$$

(b) Stationary plate temperature prescribed, moving plate thermally insulated.

$$\begin{aligned} v_z &= 0; \quad T = T_0 \text{ when } y = 0 \\ v_z &= U; \quad \frac{\mathrm{d}T}{\mathrm{d}y} = 0 \text{ when } y = h. \end{aligned} \tag{3.5}$$

We shall consider the dimensionless variables

$$W = v_z/U,\ Y = y/h, \text{ and } \phi = b(T - T_0)/G, \tag{3.6}$$

where the Griffith number is defined as

$$G = \frac{bC_0U^{2-2s}h^{2s}}{K}. \tag{3.7}$$

The Griffith number determines whether heat generation will lead to temperature differences within the melt sufficient to affect the velocity distribution locally.

The momentum equation can be integrated once to give

$$t_{zy} = \tau^* \text{ for all } y. \tag{3.8}$$

Substituting in the energy equation and non-dimensionalizing results in the second order ordinary differential equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}Y^2} + \lambda \exp\{G\phi(n-1)\} = 0, \tag{3.9}$$

where

$$n = \frac{2-2s}{1-2s} \text{ and } \lambda = \frac{bh^2\tau^{*n}}{C_0^{n-1}GK}$$

is the stress parameter. The boundary conditions to be satisfied are,

$$(a)\qquad \begin{aligned} W &= 0; \quad \phi = 0 \text{ at } Y = 0. \\ W &= 1; \quad \phi = 0 \text{ at } Y = 1. \end{aligned} \tag{3.10}$$

$$(\mathrm{b})\qquad \begin{aligned} W &= 0; \quad \phi = 0 \text{ at } Y = 0. \\ W &= 1; \quad \frac{\mathrm{d}\phi}{\mathrm{d}Y} = 0 \text{ at } Y = 1. \end{aligned} \tag{3.11}$$

Case (a)

Solving the differential equation (3.9) and using the specified boundary conditions gives λ explicitly as a function of ϕ_{max}

$$G(n-1)\lambda = 8\exp\{-G(n-1)\phi_{max}\} \times \left\{\cosh^{-1}\left[\exp\frac{G(n-1)}{2}\phi_{max}\right]\right\}^2. \quad (3.12)$$

Stationary points occur at $d\lambda/d\phi_{max} = 0$. The value of ϕ_{max} corresponding to the first stationary point of a positive λ is

$$G(u-1)\phi_{max\,crit} = 1{\cdot}187. \quad (3.13)$$

The critical value of λ is given by

$$G(n-1)\lambda = 3{\cdot}572. \quad (3.14)$$

Case (b)

The explicit relation for λ is virtually the same as for case (a)

$$G(n-1)\lambda = 2\exp\{-G(n-1)\phi_{max}\} \times \left\{\cosh^{-1}\left[\exp\frac{G(n-1)}{2}\phi_{max}\right]^2\right\}, \quad (3.15)$$

giving critical values

$$G(n-1)\phi_{max\,crit} = 1{\cdot}187$$
$$\text{and } G(n-1)\lambda_{crit} = 0{\cdot}893. \quad (3.16)$$

The values of λ_{crit} have been obtained by Turian [14] as a special case in his discussion of critical stress parameters for an Ellis fluid in plane Couette flow. Joseph [2] has obtained a similar value with $n = 2$ for a Newtonian fluid. Winter [22] has shown that for his analysis of the unsteady temperature field in plane Couette flow with the viscosity depending linearly upon temperature a critical value of βB_r arises. β is the temperature coefficient in the viscosity and B_r is a Brinkman number which incorporates a shear stress. Winter finds that the critical value for βB_r is π^2, above this value the heat generated by dissipation cannot be conducted to the walls rapidly enough; consequently the temperature increases continuously with increasing time to higher and higher values. Gruntfest [12] introduces a time ratio in his work on the unsteady temperature field which is, however, simply related to the shear stress parameter developed by Joseph [2], and in this paper. He finds that the critical value of the time ratio above which the temperature increases without limit is 0·88, and the maximum steady value of the temperature is 1·19, in agreement with this paper. For values of $\lambda > \lambda_{crit}$ no steady state solutions to the problem exist locally, while for $\lambda < \lambda_{crit}$ one value of λ can correspond to two distinct maximum temperatures in the flow. The homogeneous, self adjoint, linear comparison system which gives an estimated upper bound for λ_{crit} will be

$$\frac{d^2\phi}{dY^2} + \Lambda G(n-1)\phi = 0, \quad (3.17)$$

and the boundary conditions (3.10) and (3.11).

For the case of exponential temperature dependence referring to equation (1.23), max (ye^{-y}) occurs at $y = 1$ and takes the value e^{-1}. An estimated upper bound for λ_{crit} will then be $\lambda_{crit} = \Lambda_0/e$, where Λ_0 is the first positive eigenvalue of the linear comparison system. Putting $\Lambda G(n-1) = k^2$, the linear Sturm–Liouville system gives k as the root of $\sin k = 0$ (case (a)) and $\cos k = 0$ (case (b)).

The estimated values of λ_{crit} are

$$\text{case (a):}\quad G(n-1)\lambda_{crit} = \frac{\pi^2}{e} = 3{\cdot}632;$$
$$\text{case (b):}\quad G(n-1)\lambda_{crit} = \frac{\pi^2}{4e} = 0{\cdot}908. \quad (3.18)$$

It can be seen that the error in the estimated stress parameter is approximately 2 per cent, independent of the power law parameter n.

Poiseuille cylindrical pipe flow

The governing equations for the steady flow of viscous fluid in a cylindrical pipe with circular cross section referred to cylindrical polar coordinates (r, θ, z) are

Linear momentum

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rt_{rz}) = -\frac{\mathrm{d}t_{zz}}{\mathrm{d}z}, \tag{3.19}$$

where the z-axis is considered to be along the centre of the cylinder parallel to the generators, and

Energy

$$\frac{K}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) + t_{rz}\frac{\mathrm{d}v_z}{\mathrm{d}r} = 0. \tag{3.20}$$

The energy equation (3.20) expresses the fact that all the heat generated by internal friction within the pipe is assumed to be conducted away radially.

The only non-vanishing stress components are t_{rz} and t_{zz} given by

$$t_{rz} = 2^{-2s}C_0\,\mathrm{e}^{-b(T-T_0)}\left\{\left(\frac{1}{2}\frac{\mathrm{d}v_z}{\mathrm{d}r}\right)\right\}^{-s}\frac{\mathrm{d}v_z}{\mathrm{d}r}, \tag{3.21}$$

$$t_{zz} = -p. \tag{3.22}$$

We shall discuss the simple boundary condition $v_z = 0$; $T = T_0$ at $r = R$, the wall of the cylinder. The only difficulty that arises from consideration of a general homogeneous heat flux boundary condition is algebraic.

The momentum equation again can be integrated to give

$$t_{rz} = \frac{\mathrm{d}p}{\mathrm{d}}\frac{r}{2}, \tag{3.23}$$

where the requirement of finite velocity gradient at $r = 0$ enables the constant of integration to be set equal to zero.

Writing

$$\frac{\mathrm{d}v_z}{\mathrm{d}r} = \operatorname{sgn}\frac{\mathrm{d}p}{\mathrm{d}z}\left(\frac{\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|\left|\frac{r}{2}\right|\mathrm{e}^{b(T-T_0)}}{C_0}\right)^{\frac{1}{1-2s}} \tag{3.24}$$

in order that a negative pressure gradient corresponds to negative velocity gradient even for general non-integer values of $1/1-2s$ and substituting into the energy equation results in

$$\frac{K}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) + \left|\frac{\mathrm{d}p}{\mathrm{d}_z}\right|\frac{r}{2} \times\left\{\frac{\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|\frac{r}{2}\mathrm{e}^{b(T-T_0)}}{C_0}\right\}^{\frac{1}{1-2s}} = 0. \tag{3.25}$$

Again introduce the dimensionless variables

$$x = \frac{r}{R},\quad \phi = \frac{b(T-T_0)}{G},\quad G = \frac{bC_0R^{2s}U^{2-2s}}{K} \tag{3.26}$$

with U a characteristic velocity, for example, the mean flow velocity $Q/\pi R^2$, with Q the volumetric flow rate.

The energy equation is given in non-dimensional form as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + \lambda x^{n+1}\,\mathrm{e}^{(n-1)G\phi} = 0, \tag{3.27}$$

where

$$\lambda = \frac{b\left|\frac{\mathrm{d}p}{\mathrm{d}z}\right|^n R^{n+2}}{GK2^nC_2^{n-1}}$$

is a pressure gradient parameter. The boundary conditions are $\phi = 0$ at $x = 1$, $\mathrm{d}\phi/\mathrm{d}x = 0$ at $x = 0$ (symmetry) so that the maximum temperature occurs at the pipe centre. This equation has been solved in closed form by Martin [13], who showed that a physically realisable flow is possible only if

$$G(n-1)\lambda \leqslant \frac{(n+2)^2}{2} \tag{3.28}$$

by consideration of the roots of a quadratic equation. This is essentially equivalent to finding the first stationary point for λ considered as a function of $\phi_{\max}$.

One finds that a stationary point occurs when

$$\exp\left(-G\frac{(n-1)}{2}\phi_{\max}\right) = \tfrac{1}{2}, \tag{3.29}$$

which consequently gives

$$G(n-1)\lambda_{\mathrm{crit}} = \frac{(n+2)^2}{4}. \tag{3.30}$$

The linear comparison system for an estimated value of λ is now given by

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda G(n-1)x^{n+2}\phi = 0, \quad (3.31)$$

and

$$\phi = 0 \text{ at } x = 1, \quad \frac{d\phi}{dx} = 0 \text{ at } x = 0. \quad (3.32)$$

The substitution $x = \xi^\alpha$ reduces the differential equation to that of the zeroth order Bessel equation of the form

$$\xi^2\frac{d^2\phi}{d\xi^2} + \xi\frac{d\phi}{d\xi} + \gamma^2\xi^2\phi = 0 \quad (3.33)$$

with

$$\gamma^2 = \lambda\left(\frac{2}{n+2}\right)^2 G(n-1). \quad (3.34)$$

The eigenvalues are given by the roots of the equation

$$J_0(\gamma) = 0. \quad (3.35)$$

The smallest positive roots of this equation is

$$\gamma = 2{\cdot}405. \quad (3.36)$$

This implies that

$$G(n-1)\lambda_{\text{crit est}} = \frac{(2{\cdot}405)^2}{e}\frac{(n+2)^2}{4}. \quad (3.37)$$

Consequently

$$\frac{\lambda_{\text{crit exact}}}{\lambda_{\text{crit est}}} = \frac{2e}{(2{\cdot}405)^2} = 0{\cdot}94. \quad (3.38)$$

These results indicate that the error in taking the estimated λ_{crit} is only 6 per cent, again independent of the power law parameter n.

The two cases above should give some confidence that λ_{crit} estimated from the linear comparison equation is a reasonable upper bound for the exact λ_{crit}.

Other well known steady flows have been investigated in a similar manner. The energy equations for Couette flow between concentric cylinders, mixed drag and pressure flow between flat plates and annular flow are given by Martin [13]. The results obtained by applying the general theory in Section 1 to these flows are given in [27].

4. SCREW EXTRUSION OF POLYMER MELTS: THE FINITE DUCT

We shall consider the model used by Zamodits and Pearson [17] for full developed flow in the metering section of a single screw extruder. A detailed account of the approximations made in this model and other similar models is given in a report by Martin, Pearson and Yates [18], and Zamodits and Pearson. Further references to screw extrusion models can be found in Tadmor and Klein [19]. In the model of Zamodits and Pearson a single-start screw of outer diameter D is considered with constant channel depth h, and constant l (this implies a constant helix angle α). An unrolling procedure can be adopted if $h/D \ll 1$. This allows the effects of curvature to be neglected and the helical flow can be replaced by flow in a long shallow box. For almost all polymer melts the viscous forces generated are very large compared to those due to gravity and inertia (i.e. the Reynolds number associated with the flow is small compared to unity). The linear momentum equation can be replaced by a simple stress equilibrium equation. With this assumption it does not matter whether the screw is considered to move relative to a stationary barrel or vice-versa. Zamodits and Pearson considered the (three-sided) bottom of the channel to be stationary and the plane representing the barrel to move relative to it.

A right-handed Cartesian coordinate system is chosen, with the x axis pointing in the 'down stream', direction, i.e. parallel to the walls of the box, the y axis perpendicular to the barrel. and the z axis roughly perpendicular to the flights.

If further a wide channel approximation is used, namely the depth h is small compared with the pitch l of the screw, then the lubrication approximation (see Pearson [15]) can be used.

This implies that over most of the flow region the velocity distribution is given solely by the relative motion of the top and bottom surfaces of the channel and by the local pressure gradient grad p where $p = p(x, z)$. The problem involves only two velocity components u_x and u_z, which are taken to be functions of y only. The depth h and grad p are taken to be constant.

The effect of the flight walls is felt, in this model, solely through the restriction it places on the cross-stream mass flux

$$q_z = \int_0^h u_z \, dy. \tag{4.1}$$

For a perfectly fitting screw

$$q_z = 0. \tag{4.2}$$

Zamodits and Pearson have considered fully-developed temperature dependent solutions for which the temperature T varies only with y (i.e. local solutions). These fully-developed flows seem to be applicable only in the last few turns of screws with long metering sections.

The x momentum equation is

$$\frac{d}{dy}\left(\mu \frac{du_x}{dy}\right) = \frac{\partial p}{\partial x}. \tag{4.3}$$

The z momentum equation is

$$\frac{d}{dy}\left(\mu \frac{du_z}{dy}\right) = \frac{\partial p}{\partial z}. \tag{4.4}$$

No-slip boundary conditions imply that

$$u_x = N\pi D \cos \alpha,$$

$$u_z = N\pi D \sin \alpha \text{ at the barrel } y = h, \tag{4.5}$$

and

$$u_x = u_z = 0 \text{ at the screw } y = 0, \tag{4.6}$$

where N is the rate of revolution of the screw. The energy equation is

$$K \frac{d^2 T}{dy^2} + e_{xy} t_{xy} + e_{zy} t_{zy} = 0, \tag{4.7}$$

where t_{xy}, t_{zy} are the non-zero shear stress components and e_{xy}, e_{zy} are the non-zero components of deformation rate.

We shall consider (for definiteness) an insulated screw and a barrel maintained at a convenient specified temperature. Again homogeneous heat transfer boundary conditions can be considered.

These conditions imply that

$$\frac{dT}{dy} = 0 \text{ at } y = 0, \text{ and } T = T_0 \text{ at } y = h. \tag{4.8}$$

Using a purely viscous power law constitutive equation (see 2.1) the only non-zero shear stress components are given by

$$t_{xy} = C_0 \, e^{-b(T - T_0)} \left[\left(\frac{du_x}{dy}\right)^2 + \left(\frac{du_z}{dy}\right)^2\right]^{-s} \times \frac{du_x}{dy} = C \frac{du_x}{dy}, \tag{4.9}$$

$$t_{yz} = C_0 \, e^{-b(T - T_0)} \left[\left(\frac{du_x}{dy}\right)^2 + \left(\frac{du_z}{dy}\right)^2\right]^{-s} \times \frac{du_z}{dy} = C \frac{du_z}{dy}. \tag{4.10}$$

A first integral of the momentum equations (4.3) and (4.4) yield

$$C \frac{du_x}{dy} = \frac{\partial p}{\partial x}(y - y_1), \tag{4.11}$$

and

$$C \frac{du_z}{dy} = P_1 \frac{\partial p}{\partial x}(y - y_2), \tag{4.12}$$

where y_1, y_2 are zero stress levels and P_1 is a dimensionless pressure gradient ratio

$$\frac{\partial p}{\partial z} \Big/ \frac{\partial p}{\partial x}.$$

If the dimensionless variables

$$Y = y/h, \quad Y_1 = y_1/h, \quad Y_2 = y_2/h$$

$$U_x = \frac{u_x}{\pi D N \cos \alpha}, \qquad U_z = \frac{u_z}{\pi D N \sin \alpha} \tag{4.13}$$

together with

$$\phi = \frac{b(T - T_0)}{G}, \qquad G = \frac{bC_0(N\pi D)^{2-2s}h^{2s}}{K} \tag{4.14}$$

are introduced, the energy equation (4.7) reduces to

$$\frac{d^2\phi}{dY^2} + \lambda\{(Y - Y_1)^2 + P_1^2(Y - Y_2)^2\}^{n/2} \times e^{(n-1)G\phi} = 0, \tag{4.15}$$

where

$$\lambda = \frac{h^{n+2}b\left|\frac{\partial p}{\partial x}\right|^n}{KGC_0^{n-1}} \tag{4.16}$$

is a dimensionless pressure gradient parameter.

The temperature boundary conditions are given by

$$\frac{d\phi}{dY} = 0 \text{ at } Y = 0 \text{ and } \phi = 0 \text{ at } Y = 1. \tag{4.17}$$

Zamodits and Pearson prescribed λ then iterated on the three integral equations

$$\cos\alpha = \lambda^{n-1}\int_0^1 e^{(n-1)G\phi}(Y - Y_1)F(Y)\,dY, \tag{4.18}$$

$$\sin\alpha = \lambda^{n-1}\int_0^1 e^{(n-1)G\phi}P_1(Y - Y_2)F(Y)\,dY, \tag{4.19}$$

$$0 = \int_0^1\int_0^Y P_1(\gamma - Y_1)F(\gamma)\,e^{(n-1)G\phi}\,dY\,d\gamma, \tag{4.20}$$

where

$$F(Y) = \{(Y - Y_1)^2 + P_1^2(Y - Y_2)^2\}^{n/2}. \tag{4.21}$$

The dimensionless flow rate Q is then calculated after convergence from

$$Q = \lambda^{n-1}\int_0^1\int_0^Y e^{G(n-1)\phi}F(\gamma)(\gamma - Y_1)\,dY\,d\gamma. \tag{4.22}$$

The double valued characteristics obtained by Zamodits and Pearson must have been obtained by trying starting values for Y_1, Y_2 and P_1 in the iteration scheme.

The suggestion by Martin [20] that the flow rate Q is specified and an iteration scheme developed for λ is more appropriate to the present paper. With this scheme Martin has reproduced the double valued characteristics of Zamodits for large Griffith number. The theory in Section 1 confirms that for specified Y_1, Y_2 and P_1, a single value of $\lambda < \lambda_{\text{crit}}$ can be associated with two distinct temperature profiles which will satisfy equation (4.15). Substituting these values for Y_1, Y_2, P_1 and λ together with the distinct temperature profiles into equation (4.22) will give two distinct values for Q corresponding to a single value for λ. This suggests that mathematically it is the choice of a non-linear heat source which produces the double valued characteristics obtained by numerical computation. It is to be noted that double valued characteristics are not observed in the isothermal case. Physically the pressure gradient parameter can be increased to a certain critical level by increasing the speed of the screw say, above this level the heat generated by viscous dissipation cannot be conducted away to the barrel walls rapidly enough and no steady state condition will exist. The temperature will increase indefinitely. Zamodits and Pearson have given a physical explanation for a decrease in Q when λ increases, but they have not tried to explain why they obtain double valued characteristics.

The linear comparison system for the finite duct problem is

$$\frac{d^2\phi}{dY^2} + \lambda F(Y)(n - 1)G\phi = 0, \tag{4.23}$$

together with equations (4.17). Evaluation of λ from this system should be useful in the sense that it gives a bound on the first guess for λ in the iteration scheme. Obviously a choice of $\lambda > \lambda_{\text{crit}}$ would not provide a solution to the system. It is noticeable that Martin [20] has values of pressure gradients for which he says a solution is unobtainable. Typical numerical values of $\lambda(n - 1)G$ for specific Y_1, Y_2, P_1 and n

are given in Tables 1–3. It must be noted that the condition of no mass flow in the transverse direction implies that U_z has a stationary value in the interval $0 < Y < 1$, so that necessarily $0 < Y_2 < 1$.

Tables 1–3. Critical value of the pressure gradient parameter λ for various values of Y_1, Y_2 and P_1 ($n = 1{\cdot}31$)

Table 1. $P_1 = 0{\cdot}2$

Y_1 \ Y_2	0·10	0·30	0·50
−0·20	2·100	2·106	2·094
−0·15	2·372	2·379	2·363
−0·10	2·708	2·717	2·693
−0·05	3·126	3·137	3·099
0·00	3·652	3·661	3·601
0·05	4·298	4·306	4·211
0·10	5·046	5·068	4·932
0·15	5·862	5·925	5·740
0·20	6·673	6·800	6·656
0·25	7·314	7·529	7·251
0·30	7·556	7·844	7·558
0·35	7·248	7·530	7·308
0·40	6·516	6·745	6·605
0·45	5·638	5·804	5·729
0·50	4·808	4·922	4·884

Table 2. $P_1 = 1{\cdot}0$

Y_1 \ Y_2	0·10	0·20	0·30	0·40	0·50
−0·20	1·813	1·903	1·921	1·866	1·752
−0·10	2·207	2·352	2·379	2·285	2·101
0·00	2·715	2·954	3·000	2·837	2·535
0·10	2·288	3·704	3·797	3·523	3·033
0·20	3·704	4·425	4·650	4·235	3·503
0·30	3·797	4·650	5·108	4·644	3·754
0·40	3·523	4·235	4·644	4·385	3·626
0·50	3·033	3·503	3·754	3·626	3·183

DISCUSSION AND CONCLUSIONS

We have been able to present evidence that various models of steady flow problems with temperature dependence exhibit double-valued solutions. This means that for a given pressure gradient, say, there exists two entirely different temperature profiles. Since the energy and linear momentum equations are coupled through the temperature, the two different temperature profiles give rise to two distinct velocity profiles. Consequently one expects to obtain two different flow rates for a given pressure gradient. Furthermore, it has been shown that there exists a value of the pressure gradient (shear stress) above which no solutions to the steady problem can be found. Within the limitations of the models used this implies that for certain pressure gradients (shear stresses) no steady flow can take place. It has been pointed out, Pearson [21], that in practice a certain amount of heat leakage into the apparatus does take place. It is not to be expected that instability criteria for a complicated thermo-mechanical system can be generated from a single mechanism. Nature is far more devious than this. We are saying that if temperature were the dominant feature of the stability mechanism then the critical pressure gradients will lie in a neighbourhood of the calculated values.

Zamodits and Pearson [17] have given a physical explanation of certain regions of the characteristics they have obtained for their screw extrusion model. We have shown why their assumed theoretical model gives double-valued pressure gradient flow rate characteristics. The exponential dependence upon temperature

Table 3. $P_1 = 5{\cdot}0$

Y_1 \ Y_2	0·10	0·15	0·20	0·25	0·30	0·35	0·40	0·45	0·50
−0·20	0·567	0·647	0·722	0·777	0·795	0·767	0·709	0·620	0·539
0·00	0·602	0·695	0·786	0·857	0·883	0·848	0·766	0·667	0·572
0·20	0·617	0·720	0·825	0·909	0·942	0·904	0·810	0·698	0·593
0·40	0·609	0·711	0·816	0·904	0·942	0·909	0·817	0·704	0·598
0·60	0·585	0·678	0·772	0·849	0·883	0·871	0·778	0·679	0·583

of the viscosity and the coupling of the momentum and energy equations will lead to an ordinary differential equation which predicts such charactersistics. It is easy to believe that if internal friction produces heat at too great a rate that it cannot be conducted away at a rate sufficient to establish steady fully developed conditions. It is harder to believe that in practice for a given pressure gradient the fluid has a choice of two fully developed temperature profiles. The answer to this problem may lie in the way that the temperature profile has developed along the screw extruder. Martin *et al.* state that in practice fully developed flows will not readily arise although such solutions effectively yield upper limits on the temperature gradients that can be achieved in the melt. Also in practice convective effects caused by the presence of flight walls will be highly important. The estimation of λ from the linear comparison system could be some help in deciding on values for the first guess in the iteration scheme used to calculate pressure gradient-flow rate characteristics.

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APPENDIX

Computational Scheme

1. *General method*

We are concerned with the calculation of the smallest eigenvalue of equations of the form

$$\frac{d^2\phi}{dY^2} + \lambda F(Y)\phi = 0 \tag{A.1}$$

subject to homogeneous boundary conditions

$$\begin{aligned} h_0 \frac{d\phi}{dY}(0) + k_0\phi(0) &= 0 \\ h_1 \frac{d\phi}{dY}(1) + k_1\phi(1) &= 0. \end{aligned} \tag{A.2}$$

We construct a complete set of orthogonal functions $\{\phi_n(Y)\}$, $n = 1, 2, \ldots$ which satisfy the boundary conditions (A.2) and express the eigenfunction ϕ as an infinite series

$$\phi = \sum_{n=1}^{\infty} a_n\phi_n. \tag{A.3}$$

We then substitute a finite number of terms of (A.3) in (A.1) converting the problem into an algebraic eigenvalue problem.

2. *Construction of the orthogonal functions*

Consider the function $\phi = e^{\alpha Y} \cos wY$. This function satisfies the equation

$$\frac{d^2\phi}{dY^2} - 2\alpha\frac{d\phi}{dY} + (\alpha^2 + w^2)\phi = 0$$

and satisfies the boundary conditions (A.2) if

(i) $h_0\alpha + k_0 = 0$

(ii) $h_1 w \sin w = (h_1\alpha + k_1)\cos w$.

We assume that $h_0 \neq 0$. (If $h_0 = 0$, we construct a set of orthogonal functions in a similar manner starting with $\phi = e^{\alpha Y}\sin wY$.) We thus have $\alpha = -k_0/h_0$, and w a root of the equation $Y \tan Y = K$ where $K = (k_1h_0 - k_0h_1)/h_0h_1$ if $h_1 \neq 0$, or of the equation $\cos Y = 0$ if $h_1 = 0$. Let $w_1, w_2, \ldots$ be the positive roots of the equation $Y \tan Y = k$ (or of $\cos Y = 0$ if $h_1 = 0$) arranged in order of increasing magnitude. and let $\phi_n = e^{\alpha Y}\cos w_nY$, $n = 1, 2, \ldots$ The functions $\phi_n(Y)$ form a complete set of orthogonal functions on the interval $[0, 1]$ with respect to the weight function $e^{-2\alpha x}$ being the eigenfunctions of the Sturm–Liouville system

$$\frac{d^2\phi}{dY^2} - 2\alpha\frac{d\phi}{dY} + (\alpha^2 + \lambda)\phi = 0$$

subject to boundary conditions (A.2).

3. *Matrix formulation*

Having constructed the eigenfunctions $\{\phi_n(Y)\}$, we substitute a finite approximation for the eigenfunction

$$\phi \simeq \sum_{n=1}^{N} a_n\phi_n$$

into (A.1) and derive the equation

$$\sum_{n=1}^{N} a_n(\alpha^2 - w_n^2)\phi_n - 2\alpha\sum_{n=1}^{N} a_nw_n\sum_{m=1}^{N} b_{nm}\phi_m + \lambda_N\sum_{n=1}^{N} a_n\sum_{m=1}^{N} c_{nm}\phi_m = 0,$$

where

$$\sin w_nY = \sum_{m=1}^{\infty} b_{nm}\cos w_mY,$$
$$\phi(Y)\cos w_nY = \sum_{m=1}^{\infty} c_{nm}\cos w_mY$$

and λ_N approximates λ.

This leads to the matrix equation

$$\boldsymbol{Ea} = \lambda_N\boldsymbol{Fa} \tag{A.4}$$

where $\boldsymbol{E}$ and $\boldsymbol{F}$ are $N \times N$ matrices whose (n, m) elements are given respectively by

$$e_{nm} = \begin{cases} 2\alpha w_m b_{nm}, & m \neq n \\ 2\alpha w_n b_{nn} + w_n^2 - \alpha^2, & m = n, \end{cases}$$

$$f_{nm} = c_{nm},$$

and $\boldsymbol{a}$ is the column vector $(a_1, a_2, \ldots a_n)^T$.

If we let $\mu_N = 1/\lambda_N$ and $\boldsymbol{G} = \boldsymbol{E}^{-1}\boldsymbol{F}$, then (A.4) becomes

$$((\boldsymbol{G} - \mu_N\boldsymbol{I})\boldsymbol{a} = 0). \tag{A.5}$$

4. *Computation*

(i) The matrix $\boldsymbol{E}$ is well-conditioned in general, the size of the elements decreasing away from the main diagonal, and the method of Gaussian elimination with pivotal condensation was used to evaluate $\boldsymbol{E}^{-1}$.

(ii) Since the smallest value of λ was required, the power method was used to evaluate the largest eigenvalue of equation (A.5).

(iii) The positive roots of equation $Y \tan Y = K$ were found using an iterative method based on the equation $Y = \tan^{-1}(K/Y)$.

(iv) The computations were carried out on an I.C.L. 1903A computer at Lanchester Polytechnic. The size of N required to give three place accuracy naturally varied according to the values of the parameters, but in most cases $N = 8$ was sufficient. Mill time also varied, an average time for the calculation of a single eigenvalue being 6s.

(v) The Tables 1–3 were produced in the case of the screw extruder for ($n = 1{\cdot}31$) for various values of the parameters Y_1, Y_2 and P_1, the tabular values being those for λ/e and the boundary conditions $d\phi/dY(0) = 0$, $\phi(1) = 0$. In this case

$$F(Y) = [(Y - Y_1)^2 + P_1^2(Y - Y_2)^2]^{n/2}.$$

DISCUSSION SUR DES PARAMETRES CRITIQUES QUI PEUVENT INTERVENIR DANS L'ECHAUFFEMENT PAR FROTTEMENT POUR UN FLUIDE NON-NEWTONIEN

Résumé—Des écoulements permanents entièrement établis de fluides visqueux non-newtoniens avec génération de chaleur par viscosité sont en général représentés par des systèmes différentiels de Sturm–Liouville. On montre que sous certaines conditions, les valeurs propres non-linéaires du système prennent une valeur critique au-dessus de laquelle on ne peut obtenir de solutions locales. La théorie est appliquée à des situations spécifiques d'écoulement et dans chacun des cas une limite supérieure est évaluée pour ce paramètre critique. On montre que le terme non-linéaire de génération de chaleur est responsable des caractéristiques de débit où le gradient de pression a deux valeurs dans les modèles de systèmes d'extrudage par vis. Des valeurs calculées de la limite supérieure du paramètre critique sont données pour un modèle spécifique d'extrudeur à vis.

DISKUSSION KRITISCHER PARAMETER, DIE IN STRÖMUNGEN NICHT-NEWTONSCHER FLÜSSIGKEITEN BEI AUFHEIZUNG DURCH REIBUNG VORKOMMEN

Zusammenfassung—Vollausgebildete stationäre Strömungen von viskosen nicht-Newtonschen Flüssigkeiten mit Aufheizung durch Reibung werden gewöhnlich durch nichtlineare Sturm-Liouville-Differential-Gleichungssysteme dargestellt. Es wird gezeigt, dass unter bestimmten Bedingungen die nichtlinearen Eigenwerte des Systems einen kritischen Wert annehmen, oberhalb dessen keine lokalen Lösungen mehr angebbar sind. Die Theorie wird auf spezifische Strömungssituationen angewendet und die obere Grenze für diesen kritischen Parameter wird für jeden einzelnen Fall ausgewertet. Es wird gezeigt, dass der Ausdruck für die nichtlineare Wärmeerzeugung verantwortlich ist für die Durchflusscharakteristik mit doppelt gewertetem Druckgradient bei den Modellen für Schnecken-Extruder-Systeme. Es wurden berechnete Werte der oberen Grenze des kritischen Parameters für ein spezielles Modell eines Schnecken-Extruders angegeben.

АНАЛИЗ КРИТИЧЕСКИХ ПАРАМЕТРОВ В ПОТОКЕ НЕНЬЮТОНОВСКОЙ ЖИДКОСТИ ПРИ НАГРЕВЕ ТРЕНИЕМ

Аннотация—Полностью развитые стационарные течения чисто вязких неньютоновских жидкостей с тепловыделением за счет трения обычно описываются системой нелинейных дифференциальных уравнений типа Штурма-Лиувилля. Показано, что при определенных условиях нелинейные собственные значения системы принимают критическое значение, выше которого лакальные решения невозможны. Теория применяется к конкретным случаям течения, и в каждом случае определяется верхний предел данного критического параметра.

Покавано, что характеристики скорости потока, вызванного удвоенным (по сравнению с обычным) градиентом давления в моделях винтовых экструдерных систем, обусловлены нелинейностью типа тепловыделения. Приводятся значения верхнего предела критического параметра для конкретной модели винтового экструдера.